Abstract

In a general class of one dimensional random differential equation the convergence of the distribution function of the solution to stationary state distribution is studied. In particular it is proved the boundedness respectively the divergence of the fractional order moments of the solution below respectively above some critical exponent. This exponent is computed. In particular models it is the heavy tail exponent. When the equation is linear this exponent determines a new family of weak topologies (stronger compared to the classical one), related to the convergence to the stationary state.

MSC: 60H10, 34F05.

Keywords: Stochastic differential equation, weak convergence, heavy tail, stationary state, topological vector spaces.

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Large time behavior in random multiplicative processes.

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July 7, 2010

1 Introduction

Discrete or continuous time, one dimensional affine stochastic evolution equations (ASEE) are studied both in mathematical literature (e.g. Refs.[3, 10, 11, 14, 13]), as well as in physical literature in Refs.[16, 17, 12, 15]. This interest in ASEE comes partly from the occurrence of heavy tail (HT) in the stationary probability distribution functions (PDF), in the models of physical, economical and biological processes. The ASEE are also related to reduced models of the self-organized criticality phenomenology in plasma physics (Ref.[15]), and to the renewal processes (Refs.[3, 10, 11]).

The connection of the our approach to the linear ASEE models with HT in the stationary PDF is the following. Denote by $X_t(\omega)$ a solution of ASEE and $p_t(x) := prob_\omega(|X_t(\omega)| \geq x)$. Suppose also that the ASEE has a stationary PDF, so we denote $p(x) = \lim_{t\to\infty} p_t(x)$. We define the heavy tail exponent β_c by the asymptotic estimate $p(x) = O(x^{-\beta_c}l(x))$, where $l(x)$ is a slowly varying function.

The occurrence of the HT in the stationary PDF is related to the dynamical effect (Refs.[13, 15]) that will be studied in this article. When the stationary PDF of the solution $X_t(\omega)$ of the ASEE has HT with exponent β_c, then for $t \to \infty$ the fractional order moments $\mathbb{E}[|X_t|^p]$ remains bounded for $0 < p < \beta_c$,

respectively diverges for $p > \beta_c$. This is related to the "variance explosion" phenomena studied recently in the mathematical finance (Ref.[13]). In this article we obtain a simple, explicit formula for β_c.

Previously explicit algebraic methods for computing β_c were elaborated in some special cases: in the framework of the discrete time models in Refs. [3, 17], with i.i.d. additive and multiplicative noise, and in the case of i.i.d. additive noise and multiplicative noise modelled by a finite state Markov process in Ref.[11].

In the continuos time case, with the multiplicative noise modelled by a finite state Markov process, rigorous foundation of the computation of β_c was obtained in Ref.[10]. In Ref.[15] the multiplicative and additive noise were modelled by a superposition of Ornstein-Uhlenbeck processes. An explicit formula for β_c was derived by asymptotic methods. In all of the cases the exponent β_c is independent of the additive term.

The ASEE model equation that is considered here is a class of one dimensional random differential equation (RDE), which extends previous results from Ref.[15], by using new topological vector space methods. The additive and the multiplicative random terms in our model are stationary processes. The multiplicative term is a generalization of the stationary Gaussian processes, having a very general, possibly algebraic, correlation decay. Our results and those from Ref. [10] are complementary.

The main part of the new results are related to the convergence to stationary state in a class of linear RDE. Nevertheless, the formula for β_c was extended to a class of non linear RDE.

Define the subspace of real valued continuos functions $C_\gamma(\mathbb{R}) \subset C(\mathbb{R})$ by the condition $f \in C_\gamma(\mathbb{R}) \iff |f(x)| = o(x^{-\gamma})$. For the class of *linear models* considered here, we prove the existence of the suitable defined weak limit of the PDF of X_t, in the sense that there exists a random variable X_∞, independent of the initial conditions, such that if X_t is the solution of the RDE and $0 < \gamma < \beta_c$, then $\lim_{t \to \infty} \mathbb{E}\left[f(X_t)\right] = \mathbb{E}\left[f(X_\infty)\right]$ for all $f \in C_\gamma(\mathbb{R})$. The topology on the space of probability measures, related to this new class of weak limit, is stronger compared to the classical weak topology.

In particular we have $f(x) = |x + z|^p \in C_\gamma(\mathbb{R})$ for all $z \in \mathbb{C}$ when $0 < p < \gamma < \beta_c$, so for $t \to \infty$, we have $\mathbb{E}\left[|X_t + z|^p\right] \to \mathbb{E}\left[|X_\infty + z|^p\right]$. Moreover we prove that if $p > \beta_c$ then $\mathbb{E}\left[|X_t + z|^p\right] \to \infty$, for all but eventually one initial conditions. The expectation values, $\mathbb{E}\left[|X_t + z|^p\right]$, $\mathbb{E}\left[|X_\infty + z|^p\right]$ are related to a class of generalized L^p spaces that includes also the $p \in]0, 1[$ case. The proofs uses the geometry and topology of these general L^p spaces. In particular we prove that $X_\infty \in L^p$ for all $0 < p < \beta_c$. When heavy tail exists this property determines the heavy tail exponent β_c.

We apply these L^p methods to the study of nonlinear RDE , when the nonlinear term has a weak nonlinearity. In this nonlinear model we derive a weaker results. There exists also a critical exponent β_c, such that for $0 < p < \beta_c$ the moments $\mathbb{E}\left[|X_t + z|^p\right]$ are bounded for large t , for all initial conditions. For $p > \beta_c$ and for sufficiently large initial conditions the moments $\mathbb{E}\left[|X_t + z|^p\right]$

diverges exponentially for large time. The same formula for β_c give the demarcation line between the order of the divergent, respectively bounded fractional moments.

In all of the cases β_c is given by a simple analytic formula in term of two physically significant parameters of the multiplicative term only, suggesting some non commutative Central Limit Theorem on the affine group.

The structure of this article is the following. Section 2 provides the notations, the linear model and the main theorem. Section 3 gives a sequence of propositions leading to the proof of the main results. The study of the non linear model, by using results from Section 3 is presented in Section 4. A summary of the results is given in Section 5.

2 Description of the linear model and the results.

2.1 Notations and definitions.

The stochastic processes are defined in a fixed probability space $\{\Omega, \mathcal{F}, P\}$ with expectation value $\mathbb{E}_\omega[f(\omega)] = \mathbb{E}[f] = \int_\Omega f(\omega)dP(\omega)$. By ω will be denoted a generic element of Ω. Two driving $\mathcal{F}-$measurable stochastic processes $\{\zeta_t(\omega), \ \phi_t(\omega) : \mathbb{R} \times \Omega \to \mathbb{R}^2$, and a constant $a > 0$ (the instability threshold) defines formally the linear RDE :

$$\frac{dX_t(\omega)}{dt} = -(a + \zeta_t(\omega)) X_t(\omega) + \phi_t(\omega) \tag{1}$$

whose solution can be constructed explicitly (see below). The notations $X_t(\omega)$ or X_t, for the solutions of RDE will be reserved. Without loss of generality we impose $\mathbb{E}[\zeta_t] = 0$ and deterministic initial conditions $X_0(\omega) \equiv x_0$. The argument ω will be omitted when no confusion arises. We shall denote by Ξ the set of solutions of Equation (1).

We will preserve notation L^p also for some unusual Lebesgue spaces with $0 < p < 1$ (see Ref.[9], page 75):

Notation 1 *Let $p > 0$ and denote $\sigma_p := \min(1, p)$. We define $\|f(\omega)\|_p := (\mathbb{E}[|f|^p])^{\sigma_p/p}$ and $L^p := L^p(\Omega, \mathcal{F}, P) = \{f \mid \|f\|_p < \infty\}$. More explicitly for $p \geq 1$ we have the usual norm $\|f(\omega)\|_p := (\mathbb{E}[|f|^p])^{1/p}$, while for $0 < p \leq 1$ we have the distance to the origin given by $\|f(\omega)\|_p := \mathbb{E}[|f|^p]$.*

These complete metric vector spaces were already used in the study of probability distributions having HT (Ref.[7]).

Remark 2 *Also for $p \in (0,1)$ the topology induced by the distance $\|f - g\|_p$ and the vector space structures are compatible. This follows from the general inequality*

$$\|\alpha f(\omega) + g(\omega)\|_p \leq |\alpha|^{\sigma_p} \ \|f\|_p + \|g\|_p \tag{2}$$

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where $p > 0$ and $f(\omega)$, $g(\omega) \in L^p$. All of the L^p spaces are complete (see page 75 from Ref.[9]). It is easy to see that for $p \in (0, 1)$ the unit ball is not convex and in nonatomic cases their dual is trivial.

When $0 < p \leq 1$ the Inequality (2) results from $|a + b|^p \leq |a|^p + |b|^p$.

The final convergence results will be formulated in the terms of a subspace $C_\gamma(\mathbb{R})$ of continuos functions on $\mathbb{R}$.

Definition 3 *For any $\gamma > 0$ we define the subspace $C_\gamma(\mathbb{R})$ of the space of the continuous functions $C(\mathbb{R})$ by the condition:*

$$f \in C_\gamma(\mathbb{R}) \Leftrightarrow \lim_{|x| \to \infty} \frac{|f(x)|}{(1 + |x|)^\gamma} = 0$$

and the topology on $C_\gamma(\mathbb{R})$ by the norm

$$p_\gamma(f) := \sup_{x \in \mathbb{R}} \frac{|f(x)|}{(1 + |x|)^\gamma} \tag{3}$$

In the proofs the extension of the space of Hölder-continuous functions to the whole real line proves useful, in the study of the tail effects with extreme delocalization, i.e. $\beta_c \ll 1$:

Definition 4 *If $\alpha \in {]}0, 1]$ then $f(x) \in \mathcal{H}_\alpha \Leftrightarrow \{\exists (c_1 \geq 0)$ such that $\forall (x, y \in \mathbb{R})\ [|f(x + y) - f(x)| \leq c_1 |y|^\alpha]\}$.*

We will preserve the same notation, to define a class of functions useful in the study of the more localized regimes, when the stationary PDF has $\beta_c > 1$:

Definition 5 *If $\alpha > 1$ then $f(x) \in \mathcal{H}_\alpha \Leftrightarrow \{\exists (g_i(x) \in \mathcal{H}_1,\ c_i \in \mathbb{C}\ )\ s.t.\ f(x) = \sum_i c_i |g_i(x)|^\alpha\}$.*

Remark 6 *If $f(x) \in \mathcal{H}_\alpha$ then $|f(x)| \leq a + b|x|^\alpha$, both for $\alpha \gtreqless 1$, for some constants a, b. The exact values of the constants c_i, a, b are irrelevant.*

Remark 7 *It is elementary to check that if $z \in \mathbb{C}$, $0 < \alpha$, then $f(x) = |z + x|^\alpha \in \mathcal{H}_\alpha$.*

By using the spaces $\mathcal{H}_\alpha$, $C_\gamma(\mathbb{R})$ we define a parametrized families of weak topologies on the set of probabilistic Borel measures on $\mathbb{R}$. These topologies, with indexed by a fixed $\rho > 0$, are defined as follows:

Definition 8 *Denote $\mathcal{K}_\rho := \bigoplus_{\alpha \in]0,\rho[} \mathcal{H}_\alpha$ (finite linear combinations). For $t \to \infty$, the sequence μ_t of the Borelian measures on $\mathbb{R}$ converges to the stationary measure μ_∞ in the topology $\mathcal{T}(\mathcal{K}_\rho)$ iff $\forall f(x) \in \mathcal{K}_\rho$ we have $\lim_{t \to \infty} \int_\mathbb{R} f(x) \, d\mu_t(x) = \int_\mathbb{R} f(x) \, d\mu_\infty(x)$*

In the final formulation of the results, the asymmetry in the definition of the spaces $\mathcal{H}_\alpha$ will be removed. An apparently stronger class of topologies are given by the following

Definition 9 *For $t \to \infty$, the sequence μ_t of the Borelian measures on $\mathbb{R}$ converges to the stationary measure μ_∞ in the topology $\mathcal{T}(C_\gamma)$ iff $\forall f(x) \in C_\gamma(\mathbb{R})$ we have $\lim_{t \to \infty} \int_\mathbb{R} f(x) \, d\mu_t(x) = \int_\mathbb{R} f(x) \, d\mu_\infty(x)$.*

We observe that the $\mathcal{T}(C_\gamma)$ is strictly stronger than the weak topology.

2.2   Specification of the linear model.

Without loss of generality and for sake of simplicity, we suppose that a suitable rescaling of the time variable was performed. Consequently, the multiplicative noise ζ_t obeys the following

Condition 10 *The stochastic process ζ_t is centered and stationary. There exists a unique "diffusion constant" $D > 0$ and a set of positive constants $K_p^\pm$, all independent of the times t and s, such that for $p > 0$, $s \geq 1$ and $s - 1 \leq t \leq s$ we have:*

$$K_p^- \leq \exp\left(-D\,p^2\,s\right) \mathbb{E}\left[\exp\left\{(1-p)\,Y_s - Y_t\right\}\right] \leq K_p^+ \tag{4}$$

where

$$Y_t := \int_0^t \zeta_s \, ds \tag{5}$$

The exact values of the constants K_p^- and K_p^+ are irrelevant. This Condition is satisfied by a large class of Gaussian processes, as shown in Proposition 24.

We will prove that the large time asymptotic behavior of the solution of the RDE (1) is determined by the parameter a from Equation (1) and the constant D from Equation (4). So we introduce the following

Notation 11 *: The main results are expressed in terms of the critical exponent*

$$\beta_c := a/D \tag{6}$$

Another important quantity is $\gamma_p := \sigma_p\,D\,(p - \beta_c)$ with $p > 0$, whose sign changes at $p = \beta_c$.

On the additive noise $\phi_t(\omega)$ we impose the following

Condition 12   *1. The process $\phi_t(\omega)$ is stationary.*

2. The processes $\zeta_t(\omega)$, $\phi_t(\omega)$ are independent.

3. We have $\mathbb{E}\left[|\phi_t(\omega)|^p\right] \leq m_p < \infty$, $\forall p \in]0, \beta_1[$, where $\beta_1 > \max(1, \beta_c)$.

4. Reversibility: the equality in distribution $\phi_t(\omega) \stackrel{d}{=} \phi_{-t}(\omega)$.

From Conditions 12, item 3 results that the additive noise is allowed to have a HT with exponent larger than $\max(1, \beta_c)$.

2.3   The results, for the linear model.

Under the previous notations, Conditions 10 and 12, we have the following results, that will be demonstrated in the next sections. First we have the main Theorem and its Corollary

Theorem 13 *Let $f \in C_\gamma(\mathbb{R})$ and $0 < \gamma < \beta_c$. There exists $X_\infty(\omega) \in \bigcap_{p \in]0,\beta_c[} L^p$ such that:*

1. $f(X_\infty) \in L^1$

2. We have $\lim_{t \to \infty} \mathbb{E}[f(X_t)] = \mathbb{E}[f(X_\infty)]$. In particular if $p \in]0, \beta_c[$ and $z \in \mathbb{C}$, then $\lim_{t \to \infty} \mathbb{E}|X_t + z|^p = \mathbb{E}|X_\infty + z|^p < \infty$.

3. If $p > \beta_c$ then, except for at most a special value of the initial condition $X_{t=0}$, we have $\lim_{t \to \infty} \mathbb{E}|X_t + z|^p = \infty$.

This Theorem leads immediately to the following Corollary:

Corollary 14 *Denoting by $F_t(x)$ and $F_\infty(x)$ the cumulative PDF of X_t and of X_∞, respectively, then:*

1. *If $\gamma \in]0, \beta_c[$ and $f \in C_\gamma(\mathbb{R})$, then $\lim_{t \to \infty} \int_\mathbb{R} f(x)\, dF_t(x) = \int_\mathbb{R} f(x)\, dF_\infty(x)$. In particular, for any complex constant z we have $\lim_{t \to \infty} \int_\mathbb{R} |x + z|^\gamma\, dF_t(x) = \int_\mathbb{R} |x + z|^\gamma\, dF_\infty(x) < \infty$.*

2. *If $\gamma > \beta_c$ then except for at most a special value of the initial condition, we have $\lim_{t \to \infty} \int_\mathbb{R} |x + z|^\gamma\, dF_t(x) = +\infty$.*

The Item 1 of this Corollary states that weak convergence of the measure generated by $F_t(x)$ in the topology $\mathcal{T}(C_\gamma)$.

The previous theorem results from the following technical Lemma and its Corollary

Lemma 15 *There exists $X_\infty(\omega) \in \bigcap_{p \in]0,\beta_c[} L^p$ such that:*

1. $\{f(x) \in \mathcal{K}_{\beta_c}\} \Rightarrow \{f(X_\infty) \in L^1\}$.

2. $\{\, f(x) \in \mathcal{K}_{\beta_c}\,\} \Rightarrow \{\, \lim_{t \to \infty} \mathbb{E}[f(X_t)] = \mathbb{E}[f(X_\infty)]\}$. In particular for $p \in \,]0, \beta_c[$ and $z \in \mathbb{C}$, we have $\lim_{t \to \infty} \mathbb{E}\,|X_t + z|^p = \mathbb{E}\,|X_\infty + z|^p < \infty$.

3. If $p > \beta_c$ then, except for at most a special value of the initial condition $X_{t=0}$, we have $\lim_{t \to \infty} \mathbb{E}\,|X_t + z|^p = \infty$.

This Lemma leads immediately to the following Corollary:

Corollary 16 *Denoting by $F_t(x)$ and $F_\infty(x)$ the cumulative PDF of X_t and of X_∞, respectively, then:*

1. *If $p \in \,]0, \beta_c[$ and $f(x) \in \mathcal{H}_p$, then $\lim_{t \to \infty} \int_{\mathbb{R}} f(x)\, dF_t(x) = \int_{\mathbb{R}} f(x)\, dF_\infty(x)$. In particular for any complex constant z we have $\lim_{t \to \infty} \int_{\mathbb{R}} |x + z|^p\, dF_t(x) = \int_{\mathbb{R}} |x + z|^p\, dF_\infty(x) < \infty$.*

2. *If $p > \beta_c$ then except for at most an eventual special value of the initial condition we have $\lim_{t \to \infty} \int_{\mathbb{R}} |x + z|^p\, dF_t(x) = +\infty$.*

The Item 1 of this Corollary states that weak convergence of the measure generated by $F_t(x)$ in the topology $\mathcal{T}(\mathcal{K}_\rho)$.

We will prove that the topologies $\mathcal{T}(\mathcal{K}_\rho)$ are equivalent $\mathcal{T}(C_\gamma)$, when restricted to probabilistic measures.

Remark 17 *The convergence/divergence of the moments for $p \lessgtr \beta_c$ does not imply the existence of the HT. Indeed, if $\phi_t(\omega) = 0$ and $p \in \,]0, \beta_c[$, then $\lim_{t \to \infty} \|X_t\|_p = \|X_\infty\|_p = 0$. See Remark 23.*

Remark 18 *From the proofs will be clear that for $p < \beta_c$, when $\gamma_p < 0$ (see Notation 11) the speed of the convergence for large time of $\|X_t\|_p$ is $O(\exp(\gamma_p t))$. If $p < \beta_c$ then the speed of divergence is $\|X_t\|_p = O(\exp(\gamma_p t))$.*

3 Proof of the Theorem 13.

For technical reasons we introduce the following additional notations

Notation 19

$$A_t := \exp\left(-at - Y_t\right)$$

$$B_t := \int_0^t \phi_\tau A_t / A_\tau\, d\tau$$

$$H_t := \int_0^t \phi_\tau A_\tau\, d\tau$$

$$d(t) = \mathbb{E}[Y_t^2]/2$$

Furthermore, all the constants denoted by K_n with n integer, are independent of the time variables t, s, x, y.

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3.1 Some initial results.

Because the driving noises are classical functions, then according to the Notations 19 the rigorous, integral form of Equation (1), with the initial condition x_0, is

$$X_t = x_0 \, A_t + B_t \, . \tag{7}$$

Remark 20 *From Equation (7) and Equation (2) we get:*

$$\left| |x_0|^{\sigma_p} \|A_t\|_p - \|B_t\|_p \right| \leq \|X_t\|_p \leq |x_0|^{\sigma_p} \|A_t\|_p + \|B_t\|_p \, . \tag{8}$$

The following proposition is the key point in the difficult part of the proofs that needs L^p bounds when $0 < p < 1$. This case include also the study of the "extreme heavy tail" effects (Ref.[15]), when β_c is very small.

Proposition 21 *Let $U(\omega) \geq 0$ and $V(\omega) \geq 0$ be random variables such that $V, \, U\,V \in L^1$ and $\mathbb{E}\,[V] > 0$. Then for $p \in \,]0,1]$ we have*

$$\mathbb{E}\,[U^p \, V] \leq (\mathbb{E}\,[U\,V])^p \, (\mathbb{E}\,[V])^{1-p} \, . \tag{9}$$

Proof. This result follows from Jensen inequality applied to the concave function $x \to x^p$. Define a new expectation value $\mathbb{E}_1\,[f] := \mathbb{E}\,[fV]\,/\mathbb{E}\,[V]$.

For any concave function $g(x) : \mathbb{R} \to \mathbb{R}$ by Jensen inequality we have $\mathbb{E}_1\,[g(f)] \leq g\,\{\mathbb{E}_1\,[f]\}$. Consider now $g(x) := x^p$ and $f(\omega) = U(\omega)$. We obtain

$$\mathbb{E}\,[U^p \, V]\,/\mathbb{E}\,[V] \leq (\mathbb{E}\,[U\,V]\,/\mathbb{E}\,[V])^p$$

that leads to the Inequality (9). ■

The following Proposition results from Condition 10 by setting $s = t$:

Proposition 22 *Under the Condition 10, there exist positive constants D, K_p^+, K_p^-, K_1, and K_2 such that $\forall p \in \,]0, \infty[$ and $t \geq 0$ we have*

$$K_p^- \, \exp(p^2 \, D \, t) \leq \mathbb{E}\,[\exp(-p \, Y_t)] \leq K_p^+ \exp(p^2 \, D \, t) \tag{10}$$

$$K_2 \exp\,(t \, \gamma_p) \leq \|A_t\|_p \leq K_1 \exp\,(t \, \gamma_p) \tag{11}$$

Remark 23 *From Inequality (11) and Notation 11 results that for $0 < p < \beta_c$ we have $\|A_t\|_p \to 0$ when $t \to \infty$. In the particular case $\phi_t(\omega) = 0$, by Equation (7) we obtain also $\|X_t\|_p \to 0$.*

Proposition 24 *Suppose the process ζ_t is Gaussian, stationary, centered, with continuous realizations and has the correlation decay for $t \to \infty$*

$$C(|t|) = \mathbb{E}(\zeta_{t+\tau} \, \zeta_\tau) = \mathcal{O}(1 + |t|^{-2-\varepsilon}). \tag{12}$$

Then Condition 10 is satisfied and the Inequalities (4, 10, 11) results.

Proof. Because ζ_t is stationary, we will use a method similar to the derivation of the Taylor-Green-Kubo formula (Refs.[2, 18, 4, 5]). For the sake of completeness we adapt the proof. By using Equations (5, 12) and the Notations 19, we obtain

$$2\,d(t) = \mathbb{E}[Y_t^2] = 2\int_0^t (t-x)\,C(x)\,dx \tag{13}$$

Combined with Equation (12) it results that for $t \to \infty$ we have $d(t) = D\,t + \mathcal{O}(1)$, where

$$D = \int_0^\infty C(|\tau|)\,d\tau = \int_0^\infty \mathbb{E}[\zeta_\tau \zeta_0]d\tau \tag{14}$$

Also from Equations (12, 13) for $|x| < 1$ we have

$$d(t+x) - d(t) = \mathcal{O}(1) \tag{15}$$

As a consequence, there exists a constant K_3 such that for all $t \geq 0$ we have uniformly in t: $(D\,t - K_3) \leq d(t) := \mathbb{E}[Y_t^2]/2 \leq (D\,t + K_3)$.

Because Y_t has stationary increments we obtain $\mathbb{E}[Y_y\,Y_t] = d(t) + d(y) - d(y-t)$. With $Z = (1-p)\,Y_s - Y_t$ we get $\mathbb{E}\left[Z^2/2\right] = p^2\,d(s) + [p\,(d(t) - d(s)) + (1-p)\,d(s-t)]$.

From $|s-t| \leq 1$ and Equation (15) it results that the last bracket is uniformly bounded. Since Z is Gaussian, we obtain the Inequality (4) and by Proposition 22 the Inequalities (10, 11). ∎

Observe that by Equation (14) the constant D is related to the zero frequency component of the correlation function. We shall prove the following general

Proposition 25 *Let $p \in\ ]0, 1[$, $0 \leq y - 1 \leq x \leq y$ and suppose that*

1. *The stochastic processes Y_t, ζ_t satisfies Condition 10.*

2. *The stochastic process f_t is stationary, independent of Y_t and we have the bound uniformly in t:*
$$|\mathsf{E}[f_t]| < K_4 < \infty. \tag{16}$$

Denote
$$b_f(x, y) := \left\| \int_x^y f_t\,A_t\,dt \right\|_p \tag{17}$$

Then for some constant K_5 we have

$$b_f(x, y) \leq K_5 \exp\left(y\gamma_p\right). \tag{18}$$

Proof. Since for $p \in\ ]0, 1[$ we have $\sigma_p = p$ and $\|g\|_p = \mathbb{E}\left[|g|^p\right]$, the Equation (17) can be rewritten as $b_f(x, y) = \mathbb{E}\left[U^p\,V\right]$, where

$$V(\omega) = \exp\left[-p\,Y_y(\omega)\right]$$
$$U(\omega) = \int_x^y f_t(\omega)\,A_t(\omega)\,\exp\left[Y_y(\omega)\right]\,dt.$$

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Recalling Proposition 21 and Inequality (9) we get:

$$b_f(x,y) \leq \left(\mathbb{E}\left[U\,V\right]\right)^p \left(\mathbb{E}\left[V\right]\right)^{1-p}. \tag{19}$$

Then, on the one hand, according to Proposition 22 and Inequality (10), the term $\mathbb{E}\left[V\right]$ in Equation (19) is bounded by

$$\mathbb{E}\left[V\right] \leq K_6 \, \exp\left(p^2\,y\,D\right). \tag{20}$$

On the other hand, due to the independence of f_t and Y_t we obtain

$$\mathbb{E}\left[U\,V\right] = \int_x^y \mathbb{E}[f]\,\mathbb{E}[A_t\,\exp\left\{(1-p)Y_y\right\}]\,dt.$$

Recalling Inequality (16), and the definition of A_t, it follows that

$$|\mathbb{E}\left[U\,V\right]| \leq K_4 \int_x^y \exp(-a\,t)\mathbb{E}\left[\exp\left\{(1-p)\,Y_y - Y_t\right\}\right]\,dt$$

which, with the help of Inequality (4), reduces to

$$|\mathbb{E}\left[U\,V\right]| \leq K_7 \, \exp(-a\,y + p^2\,D\,y). \tag{21}$$

After simple calculations, recalling Inequalities (19, 20, 21), the Inequality (18) is obtained. ∎

Remark 26 *From the Condition 12, for fixed T, by symmetry and stationarity, we have an equality in distribution of the direct and reflected process $\phi_t \overset{d}{=} \phi_{T-t}$.*

From the previous Remark, it follows the following

Proposition 27 *If $f(x) \in \mathcal{H}_p$ then $\mathbb{E}[f(B_t)] = \mathbb{E}[f(H_t)]$ and in particular $\|B_t + z\|_p = \|H_t + z\|_p$, where $z \in \mathbb{C}$, $p > 0$.*

Proof. From the Definitions 4 5 results, that if $f(x) \in \mathcal{H}_p$ then $f(x)$ is continuos, hence measurable. From Remark 6 we have $|f(x)| \leq c_1 + c_2\,|x|^p$. According to the Notations 19

$$B_t = \int_0^t \phi_\tau \exp\left(-a(t-\tau) - (Y_t - Y_\tau)\right)\,d\tau$$

From Condition 10 results that the process Y_τ has stationary increments, so for fixed t we have the equality in distribution of the processes $Y_t - Y_\tau \overset{d}{=} Y_{t-\tau}$. On the other hand, from Condition 2 results that because the processes ϕ_τ, Y_τ are independent, we have the equality in distribution

$$(\phi_\tau, Y_t - Y_\tau) \overset{d}{=} (\phi_\tau, Y_{t-\tau})$$

Consequently

$$B_t \overset{d}{=} \int_0^t \phi_\tau \exp\left(-a(t-\tau) - Y_{t-\tau}\right) d\tau$$

By the change of the integration variable $\tau \to t - \tau$ and from Remark 26 results the equality in distribution

$$B_t \overset{d}{=} \int_0^t \phi_{t-\tau} \exp\left(-a\tau - Y_\tau\right) d\tau \tag{22}$$

According to the Remark 26, *for fixed t we obtain* $\phi_\tau \overset{d}{=} \phi_{t-\tau}$. By the independence of the processes ϕ_τ, ζ_τ, consequently the independence of ϕ_τ, Y_τ we have

$$(\phi_\tau, Y_\tau) \overset{d}{=} (\phi_{t-\tau}, Y_\tau)$$

So from Equation (22) results

$$B_t \overset{d}{=} \int_0^t \phi_\tau \exp\left(-a\tau - Y_\tau\right) d\tau$$

or (see Notation 19) $B_t \overset{d}{=} H_t$ which completes the proof. ■

The following Lemma and its Corollary 30 will be used in subsequent works.

Lemma 28 *Under the previous Notations 11, 19, Conditions 10 and 12, we have*

1. *If $v \geq u \geq 0$, $\gamma_p \neq 0$ and $0 < p < \beta_1$ then we have the L^p bound*

   $$\|H_v - H_u\|_p \leq K_8 \left\{\exp(u\,\gamma_p) + \exp(v\,\gamma_p)\right\} \tag{23}$$

   for some constant K_8 independent of u, v.

2. *There exists $X_\infty \in \bigcap_{0 < q < \beta_c} L^q$ such that $\lim\limits_{t \to \infty} \|H_t - X_\infty\|_p = 0$ when $p \in]0, \beta_c[$.*

3. *$\forall p \in]0, \beta_1[$ we have*

   $$\|B_t\|_p \leq K_8 \left[1 + \exp(t\,\gamma_p)\right] \tag{24}$$

Proof.

1. *If $p \in [1, \beta_1[$ then* (see Notation 19) we obtain $\|H_v - H_u\|_p \leq \int_u^v \|\phi_s A_s\|_p \, ds$. Recalling Conditions 12 items 2 and 3, it follows that

   $$\|\phi_s A_s\|_p = \|\phi_s\|_p \|A_s\|_p \leq m_p \|A_s\|_p.$$

   Consequently, by using the Inequality (11) results

   $$\|H_v - H_u\|_p \leq m_p K_1 \int_u^v \exp\left(t\,\gamma_p\right) ds$$

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which immediately lead to Equation (23).

In the "very heavy tail" case when $p \in]0,1[$ we use the Proposition 25, where we replace $f_t \equiv \phi_t$ and Equation (17). We introduce the notations $[x]$ for the integer part of x, the notations $n_v := [v-u], v_- := u + n_v$ and use the decomposition

$$\int_u^v \phi_\tau A_\tau d\tau = \left(\int_u^{u+1} + \cdots + \int_{u+n_v-1}^{u+n_v} + \int_{u+n_v}^v \right) \phi_\tau A_\tau d\tau$$

We get

$$\|H_v - H_u\|_p = b_\phi(u,v) \le b_\phi(v_-,v) + \sum_{k=1}^{n_v} b_\phi(u+k-1, u+k)$$

By using Proposition 25, Equation (18) with $f_t \equiv \phi_t$, after simple estimations we obtain Equation (23), which completes the proof.

2. Let $p \in]0, \beta_c[$. Then we have $\gamma_p < 0$ and from Equation (23) it results that if $t_n \to \infty$ then H_{t_n} is a Cauchy sequence in L^p.

   The L^p spaces, including $p \in]0,1[$, are complete (Ref.[9], page 75), therefore an $X_\infty(\omega) \in L^p$ exists such that $\|H_t - X_\infty\|_p \to 0$. Because $0 < p \le q \Rightarrow L^p \supset L^q$, it results that $X_\infty(\omega) \in \bigcap_{0<q<\beta_c} L^q$.

3. It results from Proposition 27 and Inequality (23), by setting $u = 0$, $v = t$.

∎

Lemma 29 *Let $f(x) \in \mathcal{H}_p$ fixed. Then $\Psi \in L^p \Rightarrow f(\Psi) \in L^1$. The corresponding application $L^p \to \mathbb{C}$, defined as $L^p \ni \Psi \to \mathbb{E}[f(\Psi)] \in \mathbb{C}$, is continuous.*

Proof. From $f(x) \in \mathcal{H}_p$ results $|f(x)| \le a + b|x|^p$ (see Remark 6). Hence $\Psi \in L^p \Rightarrow f(\Psi) \in L^1$.

For the proof of the continuity we consider first the case $0 < p \le 1$. Let $\Psi, \chi \in L^p$ and recall that in this case $f(x) \in \mathcal{H}_p \Rightarrow |f(x+y) - f(x)| \le c_1|y|^p$. Then it follows that

$$|\mathbb{E}[f(\Psi + \chi)] - \mathbb{E}[f(\Psi)]| \le \mathbb{E}|f(\Psi + \chi) - f(\Psi)| \le c_1 \mathbb{E}|\chi|^p = c_1 \|\chi\|_p$$

which proves the continuity in the case of the (unusual) L^p space.

In the case $p \ge 1$ we observe that according to the Definition 5 it is sufficient to prove the continuity for the set of functions of the form $f(x) = |g(x)|^p$ when $g(x) \in \mathcal{H}_1$, that generates the space $\mathcal{H}_p$ by finite linear combinations.

Let $\Psi, \chi \in L^p$ with $p \ge 1$ and recall that in this case $f(x) \in \mathcal{H}_p \Rightarrow |g(x+y) - g(x)| \le c_1|y|$. So it is sufficient to prove that

$$\|\chi\|_p \to 0 \Rightarrow \mathbb{E}[f(\Psi + \chi)] \to \mathbb{E}[f(\Psi)] \tag{25}$$

In the case $f(x) = |g(x)|^p$ the previous Equation (25) is equivalent to the condition:

$$\|\chi\|_p \to 0 \;\Rightarrow\; \|g(\Psi + \chi)\|_p \to \|g(\Psi)\|_p \tag{26}$$

Observe that from the usual $(p \geq 1)$ Hölder inequality and from $g(x) \in \mathcal{H}_1$ we obtain

$$\left| \|g(\Psi + \chi)\|_p - \|g(\Psi)\|_p \right| \leq \|g(\Psi + \chi) - g(\Psi)\|_p \leq \|c_1 \chi\|_p$$

which completes the proof for the case $p \geq 1$. ∎

Corollary 30 *Let $p \in]0, \beta_c[$ and $f(x) \in \mathcal{H}_p$. Then $\mathbb{E}[f(H_t)] \to \mathbb{E}[f(X_\infty)]$ and $\mathbb{E}[f(B_t)] \to \mathbb{E}[f(X_\infty)]$ for $t \to \infty$.*

Proof. The convergence of $\mathbb{E}[f(H_t)]$ results from Lemma 28 part 2, and Lemma 29. From Proposition 27 results $\mathbb{E}[f(B_t)] = \mathbb{E}[f(H_t)]$, which completes the proof. ∎

Remark 31 *The distribution of $X_\infty \in L^p$, with $p \in]0, \beta_c[$ is identical with $\lim_{t \to \infty} H_t := \int_0^\infty \phi_\tau A_\tau d\tau$, where the limit is in L^p. Clearly, in generic cases X_∞ is non degenerate, and for the limiting measure $\mu(x) = prob(|X_\infty(\omega)| \geq x)$, we obtain $0 < \int_0^\infty y^p d\mu(y) < \infty$ for all $p \in]0, \beta_c[$.*

3.2 Proof of the Lemma 15.

Proof. The random variable $X_\infty \in \cap_{0 < p < \beta_c} L^p$ was **constructed** in Lemma 28 part 2. Let $p \in]0, \beta_c[$ and $f(x) \in \mathcal{H}_p$. If $f(x) \in \mathcal{K}_{\beta_c}$, then it can be represented in the form $f(x) = \sum_{i=1}^n f_i(x)$ where $f_i(x) \in \mathcal{H}_{p_i}$ for $p_i \in]0, \beta_c[$ and $1 \leq i \leq n$. It is therefore sufficient to consider the case when $f(x) \in \mathcal{H}_p$ for $p \in]0, \beta_c[$.

1. It is sufficient to prove that $\{f(x) \in \mathcal{H}_p, p \in]0, \beta_c[\} \Rightarrow \{f(X_\infty) \in L^1\}$. We now use $X_\infty \in L^p$ and Lemma 29.

2. Let $t \to \infty$. It is sufficient to prove that $\{f(x) \in \mathcal{H}_p, p \in]0, \beta_c[\} \Rightarrow \{\lim_{t \to \infty} \mathbb{E}[f(X_t)] = \mathbb{E}[f(X_\infty)]\}$. From Proposition 27 it results that

$$\mathbb{E}[f(B_t)] = \mathbb{E}[f(H_t)].$$

Consequently, according to Corollary 30, we have

$$\lim_{t \to \infty} \mathbb{E}[f(B_t)] = \mathbb{E}[f(X_\infty)] < \infty.$$

We will use now Lemma 29 with $\chi = x_0 A_t$ and $\Psi = B_t$. Because for $0 < p < \beta_c$ we have $\exp(t\gamma_p) \to 0$ then, by Inequality (11) results $\|A_t\|_p \to 0$, so $\|\chi\|_p \to 0$. Consequently by Lemma 29 we obtain

$$\mathbb{E}[f(X_t)] = \mathbb{E}[f(B_t + x_0 A_t)] \to \mathbb{E}[f(X_\infty)].$$

$\mathbb{E}[f(X_t)] = \mathbb{E}[f(B_t + x_0 A_t)] \to \mathbb{E}[f(X_\infty)]$. The last remark follows from the fact that $f(x) \in \mathcal{H}_p$ for all $z \in \mathbb{C}$ if the function $f(x)$ is defined by $f(x) := |x + z|^p$.

3. Consider the case when $p > \beta_c$ *i.e.* $\gamma_p > 0$ and therefore $\exp(\gamma_p t) \to +\infty$. Suppose, ad absurdum, that there exist two initial conditions x_0 and x_0' such that the corresponding solutions $X_t(\omega)$ and $X_t'(\omega)$ of Equation (1) are bounded in L^p.

The stochastic process $Z_t(\omega) = X_t(\omega) - X_t'(\omega)$ is also bounded and it is determined by the homogenous equation. According to Equation (7), it results that

$$Z_t(\omega) = (x_0 - x_0') \, A_t(\omega)$$

and thus $\|Z_t\|_p = |(x_0 - x_0')|^{\sigma_p} \|A_t\|_p$. Then, recalling the first part of Inequality (11) we finally obtain

$$\|Z_t\|_p \geq |(x_0 - x_0')|^{\sigma_p} \, K_2 \, \exp(\gamma_p t) \tag{27}$$

Because when $p > \beta_c$ *we have.* $\gamma_p > 0$ and therefore $\exp(\gamma_p t) \to +\infty$., by Equation (27) the proof is completed.

∎

3.3 Proof of the Theorem 13.

The main point of the proof is the density of the space $\mathcal{H}_\alpha$ in the space $C_\gamma(\mathbb{R})$. For the convenience of the reader, we recall some very general definitions and a generalization by Nachbin of the Kakutani-Stone density theorem from (Ref. [8], Theorem 2) to the non compact case. See also (Ref. [6]).

3.3.1 The Nachbin approximation Lemma

We use the terminology and notations from (Ref. [6]). Let X a completely regular topological space. By $C(X)$ we denote the space of continuous real valued functions on X.

A directed set V, where $V \subset C(X)$, is a set such that for any $v_1, v_2 \in V$ there exist $\lambda > 0$ and $v \in V$ such that $v_1(x) \leq \lambda v(x)$ and $v_2(x) \leq \lambda v(x)$.

We suppose also that for any $x \in X$ there exists $v \in V$ such that $v(x) > 0$ (i.e. the set V is pointwise strictly positive). A function $f(.) \in C(X)$ is an element of the weighted space $CV_\infty(X)$ if and only if for all $v \in V$ the function $v(x)f(x)$ vanishes at infinity.

The topology on $CV_\infty(X)$ is defined by the seminorms indexed by elements of V, denoted $p_v(f)$

$$p_v(f) = \sup_{x \in X} |v(x)f(x)|$$

where $f \in CV_\infty(X)$, $v \in V$.

Recall, a lattice $L \subset C(X)$ is a set of functions closed under min and max

$$f, g \in L \Rightarrow \min\left[f(x), g(x)\right] \in L \text{ and } \max\left[f(x), g(x)\right] \in L$$

The following Lemma of Nachbin (Refs [8, 6]) will be used

Lemma 32 *(Nachbin [8]) Let X be a completely regular space, V a pointwise strictly positive set of weights, L a sublattice of $CV_\infty(X)$ and $f \in CV_\infty(X)$. Then f can be approximated by elements of L in the $CV_\infty(X)$ topology if and only if for any $x, y \in X$ and $\varepsilon > 0$ there exists $g \in L$ such that*

$$|f(x) - g(x)| < \varepsilon \tag{28}$$
$$|f(y) - g(y)| < \varepsilon \tag{29}$$

The following Corollary results

Corollary 33 *For any $0 < \alpha < \gamma$ the space $\mathcal{H}_\alpha$ is dense in $C_\gamma(\mathbb{R})$ in the topology induced by the norm from Equation (3).*

Proof. In order to apply the Lemma 32 in the our case, we consider $X = \mathbb{R}$ and the set V consists of a single function $v(x) = (1 + |x|)^{-\gamma}$. In this case the space $CV_\infty(X)$ is $C_\gamma(\mathbb{R})$ from Definition 3. First we recall Remark 6 and observe that $\mathcal{H}_\alpha$ is contained in $C_\gamma(\mathbb{R})$ when $0 < \alpha < \gamma$.

To apply Lemma 32, for any fixed $\alpha > 0$ we define the subset L of $\mathcal{H}_\alpha$ as follows: $h \in L$ if and only if there exists $\eta \in \mathcal{H}_1$ such that for any $x \in \mathbb{R}$ we have $h(x) = |\eta(x)|^\alpha$. It is easy to see that in this case $h \in \mathcal{H}_\alpha$ for any $\alpha > 0$.

Indeed, for $\alpha \geq 1$ it follows from the very definition of $\mathcal{H}_\alpha$. For $0 < \alpha \leq 1$ we have $||\eta(x)|^\alpha - |\eta(y)|^\alpha| \leq |\eta(x) - \eta(y)|^\alpha$ and now we use that $\eta \in \mathcal{H}_1$, i.e. $|\eta(x) - \eta(y)| \leq c|x - y|$. So we obtain $|h(x) - h(y)| = ||\eta(x)|^\alpha - |\eta(y)|^\alpha| \leq [c|x - y|]^\alpha$

It is easy to verify that the set L defined below is a lattice.

Indeed, suppose that $h_1, h_2 \in L \subset \mathcal{H}_\alpha$, for some fixed $\alpha > 0$. This means that they can be represented as $h_1(x) = |\eta_1(x)|^\alpha$, $h_2(x) = |\eta_2(x)|^\alpha$, with $\eta_{1,2} \in \mathcal{H}_1$. Then we have $h(x) = \min[h_1(x), h_2(x)] = |\eta(x)|^\alpha$, where $\eta(x) = \min[\eta_1(x), \eta_2(x)] \in \mathcal{H}_1$, because $\mathcal{H}_1$ is closed under min, max operations. In a similar manner we prove that L is closed under $\max[]$.

In order to use the Lemma 32 it remains to prove the restrictions given by Equations (28, 29).

Moreover, because the positive and negative part of a function from $C_\gamma(\mathbb{R})$ belongs to $C_\gamma(\mathbb{R})$ too, it is sufficient to verify that any non negative function $f(x) \geq 0$ from $C_\gamma(\mathbb{R})$ can be approximated by elements of L.

But it is easy to check that by using the family of functions $g(x) = |ax + b|^\alpha \in L$, $a, b \in \mathbb{C}$, the conditions given by Equations (28, 29) of the previous Lemma 32 are obeyed, for $f(x) \geq 0$, which completes the proof. $\blacksquare$

3.3.2 Proof of the convergence of the measures (Theorem 13)

Proof. We will use Lemma 15 and its Corollary 16. In order to complete the proof, by extending the results from Lemma 15 and Corollary 16 to Theorem 13, it is sufficient to prove that form the convergence of the measures in the weak topology $\mathcal{T}(\mathcal{K}_{\beta_c})$ it follows the convergence of the measures in all of the topologies $\mathcal{T}(C_\gamma)$ where $\gamma < \beta_c$.

In this end it is sufficient to prove that from $\lim_{t\to\infty} \int_{\mathbb{R}} f(x)\,dF_t(x) = \int_{\mathbb{R}} f(x)\,dF_\infty(x)$ for $f(x) \in \mathcal{H}_\alpha$, $\alpha \in\,]0, \beta_c[$, where $F_t(x), F_\infty(x)$ where defined in Corollary 16, it follows that for any $g(.) \in C_\gamma(\mathbb{R})$, $0 < \alpha < \gamma < \beta_c$ we have the same convergence

$$\lim_{t\to\infty} \int_{\mathbb{R}} g(x)\,dF_t(x) = \int_{\mathbb{R}} g(x)\,dF_\infty(x)$$

Denote $\nu_t(x) := F_t(x) - F_\infty(x)$. From Corollary 16 results that there exists some finite constant K_γ such that

$$\int_{\mathbb{R}} (1 + |x|)^\gamma \, |d\nu_t(x)| = K_\gamma, \ 0 < \gamma < \beta_c \tag{30}$$

We emphasize that this is the point where $\gamma < \beta_c$ is used. Let $g(.) \in C_\gamma(\mathbb{R})$, and according to the previous Corollary 33 consider the sequence $f_n(x) \in \mathcal{H}_\alpha$ of approximants $g(x)$ in the topology of $C_\gamma(\mathbb{R})$ where $0 < \alpha < \gamma < \beta_c$. So we have

$$\lim_{n\to\infty} \sup_{x\in\mathbf{R}} \left| \frac{g(x) - f_n(x)}{(1 + |x|)^\gamma} \right| = \lim_{n\to\infty} p_\gamma(g - f_n) = 0 \tag{31}$$

We have to prove that for any $\varepsilon > 0$ there exists T_ε such that it $t \geq T_\varepsilon$ then $\left| \int_{\mathbb{R}} g(x) d\nu_t(x) \right| \leq \varepsilon$. We have

$$\int_{\mathbb{R}} g(x) d\nu_t(x) = \int_{\mathbf{R}} f_n(x) d\nu_t(x) + \int_{\mathbf{R}} \frac{g(x) - f_n(x)}{(1 + |x|)^\gamma} (1 + |x|)^\gamma \, d\nu_t(x)$$

or

$$\left| \int_{\mathbb{R}} g(x) d\nu_t(x) \right| \leq \left| \int_{\mathbf{R}} f_n(x) d\nu_t(x) \right| + p_\gamma(g - f_n) \int_{\mathbf{R}} (1 + |x|)^\gamma \, |d\nu_t(x)|$$

By Equations (30, 31) we select n such that $p_\gamma(g - f_n) \leq \varepsilon/(2K_\gamma)$.

Because $f_n(x) \in \mathcal{H}_\alpha$, $\alpha \in\,]0, \beta_c[$ from Corollary 16 results that there exists T_ε such that if $t \geq T_\varepsilon$ then $\left| \int_{\mathbb{R}} f_n(x) d\nu_t(x) \right| \leq \varepsilon/2$.

This last inequality completes the proof of the convergence in the topology $\mathcal{T}(C_\gamma)$, $\gamma < \beta_c$, under the hypothesis that we have convergence in the topology $\mathcal{T}(\mathcal{K}_{\beta_c})$.

Consequently the results from Lemma 15 and its Corollary 16 can be extended to Theorem 13 and Corollary 14 which completes the proof. ∎

4   Application

We consider now a generalization of the Equation (1), containing a new non linear term $\Psi_t[X_t(\omega), \omega]$. We write the equation symbolically as

$$\frac{dX_t(\omega)}{dt} = -(a + \zeta_t(\omega)) X_t(\omega) + \Psi_t[X_t(\omega), \omega] \tag{32}$$

The driving noise terms $\zeta_t(\omega)$, $\Psi_t[X_t(\omega), \omega]$ satisfy the following restrictions.

Condition 34 *The stochastic process $\zeta_t(\omega)$ obeys Condition 10. There exists a stochastic process $\phi_t(\omega) \geq 0$ such that*

$$|\Psi_t\left[X_t(\omega), \omega\right]| \leq \phi_t(\omega) \ a.e. \tag{33}$$

and $\phi_t(\omega)$ obeys Condition 12.

In this case the existence and uniqueness of the solution is guaranteed by Theorem 221, page 65 from (Ref. [1]). The heavy tail effect is manifest now by the boundedness/divergence of the moments $\mathbb{E}\left[|X_t + z|^p\right]$ when $p \lessgtr \beta_c$:

Theorem 35 *Denote by $X_t(\omega)$ the solution of the Equation (32). Under Condition 34 we have the following behavior for $t \to \infty$.*

1. when $0 < p < \beta_c$ the fractional moment $\mathbb{E}\left[|X_t + z|^p\right]$ are uniformly bounded in t.

2. For $p > \beta_c$ and for sufficiently large values of X_0, the moments $\mathbb{E}\left[|X_t + z|^p\right]$ diverges.

Proof. In analogy to the Equation (7) and Notations 19 we have the rigorous, implicit, integral form of the Equation (32)

$$X_t = x_0\, A_t + \widetilde{B_t} \tag{34}$$

$$\widetilde{B_t} := \int_0^t \Psi_t\left[X_t(\omega), \omega\right] A_t / A_\tau \, d\tau \tag{35}$$

Because $\|f(\omega)\|_p := (\mathbb{E}[|f|^p])^{\sigma_p/p}$ (see Notation 1) it is sufficient to study the *boundedness* of $\|X_t + z\|_p$, and by Equation (2), *the boundedness of* $\|X_t\|_p$.
From Equations (34, 2) results

$$|x_0|^{\sigma_p} \left\| A_t \right\|_p - \left\| \widetilde{B_t} \right\|_p \leq \| X_t \|_p \leq |x_0|^{\sigma_p} \left\| A_t \right\|_p + \left\| \widetilde{B_t} \right\|_p \tag{36}$$

According to the Inequalities (11, 36), we have

$$|x_0|^{\sigma_p} K_2 \exp\left(t\,\gamma_p\right) - \left\| \widetilde{B_t} \right\|_p \leq \| X_t \|_p \leq |x_0|^{\sigma_p} K_1 \exp\left(t\,\gamma_p\right) + \left\| \widetilde{B_t} \right\|_p \tag{37}$$

So it is sufficient to prove (in analogy to Inequality (24)) the new bound

$$\left\| \widetilde{B_t} \right\|_p \leq K_8 \left[1 + \exp(t\,\gamma_p)\right] \tag{38}$$

Observe that because $A_t/A_\tau > 0$, from Equations (33, 35) we have the inequality

$$\left| \widetilde{B_t}(\omega) \right| \leq \int_0^t \phi_t(\omega) A_t / A_\tau \, d\tau \tag{39}$$

According to the Condition 34, in the Equation (39) we can identify

$$\int_0^t \phi_t(\omega) A_t / A_\tau \, d\tau = B_t$$

.Finally, by using Inequality (24) we obtain the Inequality (38). From Inequalities (37, 38), it results

$$\| X_t \|_p \le |x_0|^{\sigma_p} K_1 \exp(t \gamma_p) + K_8 [1 + \exp(t \gamma_p)] \tag{40}$$

respectively

$$|x_0|^{\sigma_p} K_2 \exp(t \gamma_p) - K_8 [1 + \exp(t \gamma_p)] \le \| X_t \|_p \tag{41}$$

If $p < \beta_c$ then we have $\gamma_p < 0$ and from Inequality (40) results that $\| X_t \|_p$ is bounded. If $p > \beta_c$ and if the initial conditions are sufficiently large, such that $|x_0|^{\sigma_p} > K_8/K_2$, then we have $\gamma_p > 0$ so by Inequality (41) results that $\| X_t \|_p$ diverges exponentially, which completes the proof. ■

5   Conclusions.

In a class of one dimensional random differential equations the large time behavior of the solution was studied. When the equation is linear we proved that the convergence to stationary state can be described in the framework of new class of weak topologies on the set of probability distribution of the solution. These topologies are stronger, compared to the classical weak topology. The study of the weak convergence involves in a natural way the study of *the convergence of the fractional order moments of the solution*. A critical exponent β_c was defined such that the moments of the solution, of order p remains bounded if $p < \beta_c$ and diverges, on a massive set of initial conditions, when $p < \beta_c$. When heavy tail exists then β_c is the heavy tail exponent. The speed of convergence/divergence, for large time, of the moments of order p of the solution is exponential (see Remark 18), depending on $p - \beta_c$

The strength of the new family of topologies, that describe the approach to the steady state, increases with β_c.

An important result is the exact and simple Equation (6) for β_c, in term of the parameters from the multiplicative term only. The convergence in this topology of the distribution function to the stationary distribution was proved.

A new topological vector space method was used in the proofs. By these new methods we obtained an exact formula on the critical exponent β_c also in the case of a class of nonlinear models described by Equation (32). Generalization to higher dimensions remains a challenging open problem.

6   Acknowledgements

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